

Double-Horizon Limit, AdS Geometry and Entropy Function

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Abstract

We start from a generic metric which describes four dimensional stationary black holes in an arbitrary theory of gravity and show that the AdS_2 part of the near horizon geometry is a consequence of the double-horizon limit and finiteness . We also show that the field configurations of the near horizon are determined if the same conditions are applied to the equations of motion. This is done by showing that in the double-horizon limit field equations at the horizon decouple from the bulk of the space. Solving these equations gives the near horizon field configurations. It is shown that these decoupled equations can be obtained from an action derived from the original action by applying the double-horizon condition. Our results agree with the entropy function method.

1 Introduction

It is well known that in a large class of the gravity theories coupled to a number of scalars and gauge fields, the value of the scalars at the horizon of the extremal black holes is independent of the values at the large distance. This phenomenon called attractor mechanism, has been first shown for the supersymmetric theories [1]-[3] and was later studied and proved for the non-supersymmetric cases [4]. Application of the attractor mechanism in string theory and its importance in the counting of the states of black holes is discussed in [5] and [6].

A remarkable progress in understanding of the attractor mechanism came with the works of *Sen* who introduced the *entropy function* method [7]-[8]. He showed that not only scalar fields but all parameters of the near horizon of an extremal black hole is fixed by extremizing a function, called the entropy function, which is evaluated near the horizon. Sen's method resulted in a generalized attractor mechanism and was shown to work in a number of cases.

In the entropy function method, one starts with characterization of the extremal black holes with their near horizon geometry. It is taken to have an AdS_2 factor. We would like to address the question of deeper physical properties that leads to such geometric characterization in most known cases. It is also of our interest to examine whether one can find similar method for the non-extremal cases which do not have AdS_2 space in their near horizon geometry. There has been several attempts to answer these questions [9]- [11].

In this paper we show that both questions can be approached from a single point of view which is imposing finiteness on the physical quantities at the horizon of the double-horizon black holes. For most of the known four dimensional black hole solutions, extremality coincides with the double-horizon limit, i.e. the radii of the event inner and outer horizons coincide. This results in zero surface gravity on the horizon and zero temperature for the black hole. Moreover, the singularities at the horizons are coordinate singularities and can be removed by coordinate transformation, thus we expect that the scalars constructed by the metric are finite at the horizons.

The physical reason for the attractor mechanism is the infinite distance to the double horizon that prevents the information to affect the horizon physics [12]. This distance from an arbitrary point to a simple horizon is finite and hence allows the bulk information reach the horizon and therefore blocks the attractor mechanism. The double-horizon property is essential in the divergence of the distance in the first and finiteness of the latter.

This work follows and completes the work done in [9]. In section two, we start from a generic metric which describes four dimensional stationary black holes. This form is completely general and does not depend on any particular theory. Then we explore the consequences of the finiteness of scalars constructed from the metric. The finiteness is

justified since the singularities of the horizon must be removable. We demand the finiteness of R , $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ and find certain restrictions on the components of the metric. The double-horizon condition imposes further restrictions. Using these restrictions and following the method of [13], we obtain the near horizon geometry of the stationary black holes which is the starting point of [14] in generalizing the entropy function method to the rotating black holes. Our result clearly shows that the *AdS* near horizon geometry is a consequence of the finiteness at the horizon and the double-horizon limit.

In section three we use our results in [9] and the finiteness assumption and the double-horizon condition on the equations of motion to cast the equations to a set of equations that are decoupled from the bulk. We do this by starting from the general ansatz for the metric which describes both the extremal and non-extremal black holes. The decoupling occurs due to vanishing of all r -derivative terms. First we show the decoupling of the dynamics in Einstein gravity coupled to a number of scalars and abelian gauge fields and then extend it to $f(R)$ gravities. For $f(R)$ gravities we use the technique of the equivalence of these theories with Einstein-Scalar gravity [18]. These equations are non-linear ordinary differential equations with respect to θ . The solution(s) to these equations specifies the field configuration(s) at the near horizon. However concluding the attractor mechanism from this decoupling is possible if we prove that our equations have a unique solution.

Our analysis also shows that if we do not apply the double-horizon limit, the field equations at the horizon do not decouple from the bulk and thus we can not expect an attractor mechanism for the case of simple horizon. In particular we observe that it is not possible to reproduce the dynamics on the simple horizon by extremizing a function which is only defined at the horizon.

Recently [15]-[17] has found examples of non-extremal black holes for which the entropy function method works. These results are not in contradiction with ours, since the original metric which describes black holes in those cases are not the same as one we consider in this work.

Our method for dealing with the rotating black holes is the reduction of ϕ -coordinate in Kaluza-Klein style. Because of axisymmetry of the stationary black holes, parameters of the metric do not have ϕ dependence. Moreover ϕ is a periodic coordinate, therefore similar to the Kaluza-Klein reduction, we can reduce this direction and find a three dimensional gravity theory in which apart from the original scalar and gauge fields additional scalar and gauge fields origination from components of four dimensional metric and fields exist. The interesting point is that the charge of the new gauge field is exactly the angular momentum of the four dimensional black hole. This technique simplifies the derivation and the form of the decoupled equations.

In section four we establish the relation of the equations to the entropy function method. We reproduce the decoupled equations by extremizing a function at the horizon. The difference with entropy function method is that this function is calculated using the general form for the fields of the theory and imposing the double-horizon and finiteness conditions.

This results show that the possibility of derivation of the equations of motion by extremizing a function at the horizon, is a direct consequence of the double-horizon limit. We also investigate the result of our assumed conditions on the the Wald's entropy formula. Following the method of [14] we find the results previously obtained by the entropy function method. The authors of reference [19] have used the zero surface gravity property of the extremal black holes to simplify the Wald formula for static solutions. Their results are in agreement with our proposal that the decoupled physics of the horizon is a consequence of the double-horizon limit.

2 Double-Horizon limit and AdS geometry

We consider 4-dimensional stationary black holes with axial symmetry. They are described by the generic metric

$$ds^2 = -a(r, \theta) dt^2 + \frac{b(r, \theta)}{S(r)} dr^2 + e(r, \theta) d\theta^2 + f(r, \theta) d\phi^2 + 2c(r, \theta) dt d\phi \quad (2.1)$$

where $a(r, \theta)$, $b(r, \theta)$, $c(r, \theta)$, $e(r, \theta)$ and $f(r, \theta)$ are assumed to be regular functions. Event horizons are located at $r = r_H$ where $S(r_H) = 0$. In writing this metric, it is assumed that event horizons are Killing horizons too. This is a general form based on the assumed symmetries. At this stage we consider only the symmetry which is a reflection of the topology of the horizon in four dimension and no particular theory of the gravity is assumed in this part. Therefore the result is applicable to a wide class of theories . We consider the cases that $S(r)$ has non-zero roots and exclude the naked singularities.

Let us first impose the finiteness assumption on the determinant of this metric. The determinant g is a coordinate dependent quantity. The quantity which is invariant and coordinate independent is $\Delta V = \sqrt{-g} d^4x$. If we assume that the volume of any finite neighborhood near the horizon is finite then from finiteness of d^4x we conclude that $\sqrt{-g}$ is also finite. The determinant is,

$$g = -\frac{b(r, \theta) e(r, \theta) \left(a(r, \theta) f(r, \theta) + c(r, \theta)^2 \right)}{S(r)} \quad (2.2)$$

Finiteness requires,

$$a(r, \theta) f(r, \theta) + c(r, \theta)^2 = S(r) v(r, \theta) \quad (2.3)$$

where $v(r, \theta)$ is a regular function.

Finding $a(r, \theta)$ from (2.3) and substituting it in (2.1) and redefining new regular functions, we obtain,

$$ds^2 = A(r, \theta) \left(-\frac{S(r)}{B(r, \theta)} dt^2 + \frac{dr^2}{S(r)} \right) + E(r, \theta) d\theta^2 + F(r, \theta) \left(d\phi + C(r, \theta) dt \right)^2 \quad (2.4)$$

Using this metric we can calculate scalars such as R , $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. Assuming that the singularities of the horizons are coordinate singularities, we conclude that these scalars are all finite there. However, computations of these scalars show that they include terms that have $S(r)$ factor in their denominator which will diverge unless the nominators also develop similar factors. Investigation of all these terms shows that some have $\frac{\partial}{\partial\theta}B(r, \theta)$ and some $\frac{\partial}{\partial\theta}C(r, \theta)$ factor in their nominator. If these derivatives are proportional to $S(r)$ then all of them become finite at the horizons. Thus finiteness imposes the conditions,

$$\left. \frac{\partial}{\partial\theta}C(r, \theta) \right|_{r=r_H} = 0, \quad \left. \frac{\partial}{\partial\theta}B(r, \theta) \right|_{r=r_H} = 0 \quad (2.5)$$

This means that these two functions B and C must have the forms,

$$B(r, \theta) = S(r)B_1(r, \theta) + B_2(r) \quad (2.6)$$

$$C(r, \theta) = S(r)C_1(r, \theta) + C_2(r) \quad (2.7)$$

For our argument it is sufficient to take the first power of $S(r)$ in the right hand side. One can absorb higher powers in the functions $B_1(r, \theta)$ and $C_1(r, \theta)$. The regularity of the derivatives of $B(r, \theta)$ and $C(r, \theta)$ at the horizon do not allow that power of $S(r)$ be less than one. It is obvious that at the horizons $B(r, \theta)$ and $C(r, \theta)$ are constants and do not have θ -dependence. We show their values at the outer horizon by B and C .

For black holes with the double-horizon where the inner and outer horizons coincide, we have

$$S(r) = (r - r_H)^2 \quad (2.8)$$

where r_H is the radius of the horizons.

Using (2.7) and applying the double-horizon condition, it is not difficult to see that α defined as

$$\alpha \equiv \left. \frac{\partial}{\partial r}C(r, \theta) \right|_{r=r_H} \quad (2.9)$$

is a constant .

Now we are ready to find the near horizon geometry of the double-horizon black holes. First, we expand the functions specifying the metric around r_H in terms of $r - r_H$. Using (2.5) and (2.9), we get

$$\begin{aligned}
A(r, \theta) &\approx A(\theta) + \left. \frac{\partial A}{\partial r} \right|_{r=r_H} (r - r_H), \\
B(r, \theta) &\approx B + \left. \frac{\partial B}{\partial r} \right|_{r=r_H} (r - r_H), \\
C(r, \theta) &\approx C + \alpha(r - r_H), \\
E(r, \theta) &\approx E(\theta) + \left. \frac{\partial E}{\partial r} \right|_{r=r_H} (r - r_H), \\
F(r, \theta) &\approx F(\theta) + \left. \frac{\partial F}{\partial r} \right|_{r=r_H} (r - r_H).
\end{aligned} \tag{2.10}$$

The near horizon coordinates $(\hat{t}, \hat{r}, \theta, \hat{\phi})$ are defined as [13];

$$\begin{aligned}
t &= \frac{\sqrt{B}}{\lambda} \hat{t} \\
r &= r_H + \lambda \hat{r} \\
\phi &= \hat{\phi} - C \frac{\sqrt{B}}{\lambda} \hat{t}
\end{aligned} \tag{2.11}$$

in the limit $\lambda \rightarrow 0$.

Substituting (2.10) in (2.4) and using (2.11), we obtain the metric in terms of the near horizon coordinates,

$$ds^2 = A(\theta) \left(-\hat{r}^2 d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} \right) + E(\theta) d\theta^2 + F(\theta) \left(d\hat{\phi} + \Sigma \hat{r} d\hat{t} \right)^2 \tag{2.12}$$

where

$$\Sigma = \sqrt{B} \alpha \tag{2.13}$$

In this coordinate system which is suitable for the near horizon, $g_{\theta\theta}$ component of the metric has only θ dependence. This results a reparametrization freedom which allows us to fix $g_{\theta\theta}$ up to a constant by defining $\hat{\theta}$ as

$$\hat{\theta} = \frac{1}{\Gamma} \int_0^\theta d\theta' \sqrt{E(\theta')} \tag{2.14}$$

where

$$\Gamma = \frac{1}{\pi} \int_0^\pi d\theta' \sqrt{E(\theta')} \tag{2.15}$$

It is clear that $\hat{\theta}$ goes between 0 to π . Near horizon geometry finally takes the form

$$ds^2 = A(\hat{\theta}) \left(-\hat{r}^2 d\hat{t}^2 + \frac{d\hat{r}^2}{\hat{r}^2} \right) + \Gamma^2 d\hat{\theta}^2 + F(\hat{\theta}) \left(d\hat{\phi} + \Sigma \hat{r} d\hat{t} \right)^2 \quad (2.16)$$

The main point is that (2.16) is a result of the finiteness at the horizon and the double-horizon condition. This result is general and does not depend on the action of the theory. It is also independent of the asymptotic behavior of the metric. Hence it can be applied even to metrics embedded in non asymptotically flat space. (2.16) is the starting point of [14] in generalizing entropy function method for the rotating black holes. In [14] it is taken as the beginning point for the rest of the argument but we have shown it as a result of physically simple assumptions. This shows that AdS_2 near horizon geometry which is used in entropy function method as the definition of the extremal black holes is a consequence of the finiteness and the double-horizon. In the next sections we will show that in order to find the parameters of the near horizon it is not necessary to go to the near-horizon geometry and apply entropy function method. Starting from the metric (2.4) and writing the equations at the horizon and applying the double-horizon condition, we can find the parameters of the near horizon geometry directly.

Another point is that values of B and C at the horizon are absorbed in the definition of \hat{t} and $\hat{\phi}$. This means that there is a reparametrization freedom at the horizon and the values of B and C at the horizon can not be determined by equations of motion and are not physical. We can only find Σ which is sufficient for specifying the near horizon geometry. This will be clear in the next sections where by using double-horizon condition we get a set of decoupled equations at the horizon for $A(\hat{\theta})$, $F(\hat{\theta})$, Γ and Σ .

3 At the Horizon of Double-Horizon Black Holes

Analysis of the previous section shows the importance of the finiteness and the double-horizon condition in finding the near horizon geometry of the double-horizon black holes. Furthermore, in this section we would like to show that these assumptions are enough to fix the field configuration of the near horizon geometry. The interesting point here is that this fixing is possible without directly using the near horizon field configurations. It is done by applying double-horizon limit on the field equations come from variation of a generic action and using generic ansatz for the field content of the theory. The result applies to a large class of theories and is independent of the details of the dynamics in the bulk.

We first consider rotating charged black holes in a theory of gravity with scalars Φ_I ($I =$

1, 2, ...) and abelian gauge fields $A^{(K)}$ ($K = 1, 2, \dots$) described by the action

$$S = \frac{1}{\kappa^2} \int d^4x \sqrt{-\hat{g}} \left(\hat{R} - h_{IJ}(\Phi) \partial_\mu \Phi_I \partial^\mu \Phi_J - w_{KL}(\Phi) F_{\mu\nu}^{(K)} F^{(L)\mu\nu} - V(\Phi) \right) \quad (3.1)$$

where $w_{KL}(\Phi)$ determines the coupling of the scalars to the gauge fields and $V(\Phi)$ is a potential term for the scalars. After analyzing this case we will consider more general theories. As shown in the previous section the form of the metric is

$$d\hat{s}^2 = A(r, \theta) \left(-\frac{S(r)}{B(r, \theta)} dt^2 + \frac{1}{S(r)} dr^2 \right) + E(r, \theta) d\theta^2 + F(r, \theta) \left(d\phi + C(r, \theta) dt \right)^2 \quad (3.2)$$

where

$$S(r) = (r - r_+)(r - r_-) \quad (3.3)$$

r_+ and r_- are the radii of the outer and inner horizons. It is clear that for the case of double-horizon black holes i.e. $r_+ = r_-$, not only $S(r)$ but also $\frac{dS(r)}{dr}$ vanishes at the horizons and for all cases $\frac{d^2 S(r)}{dr^2} = 2$.

Using axial symmetry of the rotating stationary black holes, we choose following ansatz for the scalars and gauge fields:

$$\Phi_I = \Phi_I(r, \theta) \quad (3.4)$$

$$A_\mu^{(K)} dx^\mu = A_t^{(K)}(r, \theta) dt + A_\phi^{(K)}(r, \theta) d\phi \quad (3.5)$$

It is assumed that this black hole has angular momentum J , electric charges $Q^{(K)}$ and magnetic charges $P^{(K)}$ defined through the relation

$$P^{(K)} = \int d\theta d\phi F_{\theta\phi}^{(K)} = 2\pi \left(A_\phi^{(K)}(\pi) - A_\phi^{(K)}(0) \right) \quad (3.6)$$

As we saw in the previous section, requirement of the finiteness of the scalars constructed by the metric imposes conditions (2.5) on the functions $B(r, \theta)$ and $C(r, \theta)$.

Now we demand that at the horizons, not only the scalars constructed from the metric but also all the terms that appear in the action to be finite. It is possible to consider (3.1) as the KK-reduction of a higher dimensional pure gravity which only has $\sqrt{-g^{(d)}} R^{(d)}$ term. The singularity of the horizons in this higher dimensional picture is a coordinate singularity too and therefor $R^{(d)}$ must be finite. After the reduction, this term breaks to three terms of the 4d action (3.1) and thus sum of these terms must be finite at the horizons. However, all of these terms must be separately finite. The R term is finite since the singularity is coordinate singularity. The other two terms are both positive and hence each of them must be separately finite. h_{IJ} and w_{KL} are also finite since are reduced from the higher dimensional regular metric components. Finiteness of $F_{\mu\nu} F^{\mu\nu}$ requires that

$$F_{\theta t}^{(K)}(r, \theta) - C(r, \theta) F_{\theta\phi}^{(K)}(r, \theta) \Big|_{r=r_H} = 0 \quad (3.7)$$

Using (2.5) and (3.5) this condition takes the form:

$$\left. \frac{\partial}{\partial \theta} \left(A_t^{(K)}(r, \theta) - C(r, \theta) A_\phi^{(K)}(r, \theta) \right) \right|_{r=r_H} = 0 \quad (3.8)$$

Therefore the left hand side is proportional to $S(r)$ and we can write

$$\frac{\partial}{\partial \theta} \left(A_t^{(K)}(r, \theta) - C(r, \theta) A_\phi^{(K)}(r, \theta) \right) = S(r) f^{(K)}(r, \theta) \quad (3.9)$$

where $f^{(K)}(r, \theta)$ is a regular function¹. By integrating both sides with respect to θ we obtain

$$A_t^{(K)}(r, \theta) - C(r, \theta) A_\phi^{(K)}(r, \theta) = S(r) f_1^{(K)}(r, \theta) + f_2^{(K)}(r) \quad (3.10)$$

It is obvious from this relation that $\beta^{(K)}$'s defined as

$$\beta^{(K)} = \left. \frac{\partial}{\partial r} \left(A_t^{(K)}(r, \theta) - C(r, \theta) A_\phi^{(K)}(r, \theta) \right) \right|_{r=r_H} \quad (3.11)$$

are constant for the double-horizon black holes. It is not difficult to see that in the near horizon coordinate (2.11), we have

$$F_{\hat{r}\hat{t}}^{(K)} = \sqrt{B} \beta^{(K)} \quad (3.12)$$

thus our conditions guarantee that this component of the field strength is constant in the near horizon coordinate.

In dealing with the rotating black holes it is easier to apply a reduction similar to KK reduction in the ϕ -direction. The axisymmetry of the rotating black hole background guarantees that none of the metric components depends on ϕ which allows such reduction. Since this direction is periodic with period of 2π , we can look at it as a compact direction with unit radius of compactification. The four dimensional rotating black hole will be converted to a non-rotating but charged black object represented in this new three dimensional picture. This black object is the source of the new scalar fields and a new gauge fields. Angular momentum of the 4d rotating black hole will be charge of the new 3d black object.

Let us define

$$\begin{aligned} M(r, \theta) &\equiv F(r, \theta) A(r, \theta) \\ N(r, \theta) &\equiv F(r, \theta) E(r, \theta) \end{aligned} \quad (3.13)$$

Reducing the action (3.1) in the ϕ -direction and changing to Einstein frame, we obtain

$$S = \frac{2\pi}{\kappa^2} \int d^3x \sqrt{-g} \left(R - H^{ij}(\Phi) \partial_\mu \Phi_i \partial^\mu \Phi_j - W_{ab}(\Phi) F_{\mu\nu}^{(a)} F^{(b)\mu\nu} - U(\Phi) \right) \quad (3.14)$$

¹The power of $S(r)$ in the right hand side of (3.9) can be n ($n \geq 1$). We absorb $S(r)^{n-1}$ in $f^{(K)}(r, \theta)$. This power can not be less than one since makes $F_{\mu\nu} F^{\mu\nu}$ infinite.

where the three dimensional metric takes the form,

$$ds^2 = M(r, \theta) \left(-\frac{S(r)}{B(r, \theta)} dt^2 + \frac{dr^2}{S(r)} \right) + N(r, \theta) d\theta^2 \quad (3.15)$$

The scalars are given as

$$\Phi = \begin{pmatrix} -\frac{1}{2} \ln F(r, \theta) \\ A_\phi^{(K)} \\ \Phi_I(r, \theta) \end{pmatrix} \quad (3.16)$$

The first two components are from the $\phi\phi$ metric component and ϕ component of the gauge potential.

The three dimensional gauge potentials are

$$\mathbf{A}_t = \begin{pmatrix} C(r, \theta) \\ A_t^{(K)}(r, \theta) - C(r, \theta) A_\phi^{(K)}(r, \theta) \end{pmatrix} \quad (3.17)$$

and the moduli metric is

$$\mathbf{H} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{2}{F(r, \theta)} w_{KL} & 0 \\ 0 & 0 & h_{IJ} \end{pmatrix} \quad (3.18)$$

The gauge field couplings are given by

$$\mathbf{W} = F(r, \theta) \begin{pmatrix} \frac{1}{4} F(r, \theta) + w_{KL} A_\phi^{(K)} A_\phi^{(L)} & A_{\phi(L)} w^{LK} \\ w_{KL}(\Phi) A_\phi^{(L)} & w_{KL} \end{pmatrix} \quad (3.19)$$

and the potential is

$$U(\Phi) = \frac{1}{F(r, \theta)} V(\Phi) \quad (3.20)$$

It is seen that only the t-component of the new gauge fields is non-zero. Using (2.5) and (3.8), we see that at the horizon $F_{\theta t}$ is zero for the both gauge potentials. From the definitions (2.9) and (3.11), we deduce that in the case of the double-horizon we have,

$$\mathbf{F}_{rt} \Big|_{r=r_H} = \begin{pmatrix} \alpha \\ \beta^{(K)} \end{pmatrix} \quad (3.21)$$

In the three dimensional picture $C(r, \theta)$ becomes a component of the new gauge field. Because of the gauge freedom, it is not possible to determine this function by using the equations of motion. This gauge fixing freedom of 3d picture is a consequence of the reparametrization freedom of the original four dimensional theory.

Another subtle point in the new 3d theory is that coordinate θ goes from 0 to π and does not cover the complete 3d space. In order to solve this problem, we use the axial symmetry of the original solution and extend the new theory to the whole of the space by demanding that

$$X(\theta) = X(2\pi - \theta) \quad (3.22)$$

where X stands for any field of the 3d theory and θ covers the complete cycle between 0 to 2π . In this form the three dimensional theory is on the whole of R^3 . Requirement of the smoothness of the solution at the poles i.e $\theta = 0, \pi$ implies ,

$$\left. \frac{\partial X}{\partial \theta} \right|_{\theta=0, \pi} = 0 \quad (3.23)$$

Now we want to impose the double-horizon condition on the equations of motion. Writing equations of motion at the horizon is equivalent to removing all the terms which have $S(r)$ factor. Imposing the double-horizon condition is done by setting $\frac{dS(r)}{dr}$ factor to zero. Variation of the metric in (3.14) gives:

$$R_{\mu\nu} - H^{ij}(\Phi) \partial_\mu \Phi_i \partial_\nu \Phi_j = W_{ab}(\Phi) \left(2g^{\rho\sigma} F_{\mu\rho}^a F_{\nu\sigma}^b - g_{\mu\nu} F_{\rho\sigma}^a F^{b\rho\sigma} \right) + g_{\mu\nu} U(\Phi) \quad (3.24)$$

Setting $S(r)$ and $\frac{dS(r)}{dr}$ to zero in this equation one obtains equations,

$$4N^2(\theta) + 2N(\theta) M''(\theta) - N'(\theta) M'(\theta) + 4N^2(\theta) M(\theta) U(\phi) = 0 \quad (3.25)$$

$$4M(\theta)N(\theta) + \left(M'(\theta) \right)^2 = 2M^2(\theta) H^{ij}(\Phi) \Phi'_i \Phi'_j + 4N(\theta) V_e(\Phi) - 2N(\theta) M^2(\theta) U(\Phi) \quad (3.26)$$

where

$$V_e(\Phi) = B(W_{11} \alpha^2 + W_{1K} \alpha \beta^{(K)} + W_{K1} \alpha \beta^{(K)} + W_{KL} \beta^{(K)} \beta^{(L)}) \quad (3.27)$$

and derivatives are respect to θ . Note that the $R_{\mu\nu}$ equations have resulted only the two above equations. The equations of motion from the variation of the scalars are;

$$\frac{2}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} H^{ij}(\Phi) \partial^\mu \Phi_j \right) = \frac{\delta H^{kj}(\Phi)}{\delta \Phi_i} \partial_\mu \Phi_k \partial^\mu \Phi_j + \frac{\delta W_{ab}(\Phi)}{\delta \Phi_i} F_{\mu\nu}^a F^{b\mu\nu} + \frac{\delta U(\phi)}{\delta \Phi_i} \quad (3.28)$$

which at the horizon of the double-horizon black holes take the form;

$$\frac{M(\theta)}{\sqrt{N(\theta)}} \left(\frac{M(\theta)}{\sqrt{N(\theta)}} H^{ij}(\Phi) \Phi'_j \right)' = \frac{M^2(\theta)}{2N(\theta)} \frac{\delta H^{jk}(\Phi)}{\delta \Phi_i} \Phi'_j \Phi'_k - \frac{\delta V_e(\Phi)}{\delta \Phi_i} + \frac{1}{2} M^2(\theta) \frac{\delta U(\phi)}{\delta \Phi_i} \quad (3.29)$$

Finally the variation of the gauge fields gives;

$$\partial_\mu (\sqrt{-g} W_{ab}(\Phi) F^{b\mu\nu}) = 0 \quad (3.30)$$

The only non-trivial equation in (3.30) is when $\nu = t$. Using (3.17) in this equation and integrating with respect to θ in the interval $[0, \pi]$ gives;

$$\frac{\partial}{\partial r} \left(\int_0^\pi d\theta \sqrt{-g} g^{rr} g^{tt} W_{ab}(\Phi) F_{rt}^b \right) + \left[\sqrt{-g} g^{\theta\theta} g^{tt} W_{ab}(\Phi) F_{\theta t}^b \right]_0^\pi = 0 \quad (3.31)$$

Using (3.23), $F_{\theta t} = 0$ at $\theta = 0, \pi$, thus the second term vanishes. The first term introduces a constant which for $a = 1$ is proportional to the angular momentum and for $a = K + 1$ to the charge $Q^{(K)}$ of the 4-dimensional black hole. At the horizon it takes the form,

$$\sqrt{B} \int_0^\pi d\theta \frac{\sqrt{N(\theta)}}{M(\theta)} (W_{11}\alpha + W_{1K}\beta^{(K)}) = -2J \quad (3.32)$$

$$\sqrt{B} \int_0^\pi d\theta \frac{\sqrt{N(\theta)}}{M(\theta)} (W_{K1}\alpha + W_{KL}\beta^{(L)}) = 2Q^{(K)} \quad (3.33)$$

where J and $Q^{(K)}$ are the angular momentum and charges of the black hole. The coefficients of proportionality at the right hand sides are fixed by use of the known Kerr-Newman solution.

Defining

$$\Sigma = \sqrt{B} \begin{pmatrix} \alpha \\ \beta^{(K)} \end{pmatrix}, \quad (3.34)$$

the equations (3.25), (3.26), (3.29), (3.32) and (3.33) take the following compact forms,

$$4N^2(\theta) + 2N(\theta)M''(\theta) - N'(\theta)M'(\theta) + 4N^2(\theta)M(\theta)U(\Phi) = 0 \quad (3.35)$$

$$4\frac{N(\theta)}{M(\theta)} + \left(\frac{M'(\theta)}{M(\theta)}\right)^2 = 2(\Phi')^T \mathbf{H} \Phi' + 4\frac{N(\theta)}{M^2(\theta)} \Sigma^T \mathbf{W} \Sigma - 2N(\theta)U(\Phi) \quad (3.36)$$

$$\frac{\sqrt{N(\theta)}}{M(\theta)} \left(\frac{M(\theta)}{\sqrt{N(\theta)}} (\mathbf{H} \Phi')_i \right)' = \frac{1}{2} (\Phi')^T \frac{\delta \mathbf{H}}{\delta \Phi_i} \Phi' - \frac{N(\theta)}{M(\theta)^2} (\Sigma)^T \frac{\delta \mathbf{W}}{\delta \Phi_i} \Sigma + \frac{N(\theta)}{2} \frac{\delta U(\Phi)}{\delta \Phi_i} \quad (3.37)$$

$$\int_0^\pi d\theta \frac{\sqrt{N(\theta)}}{M(\theta)} \mathbf{W} \Sigma = 2\mathbf{Q} \quad (3.38)$$

where

$$\mathbf{Q} = \begin{pmatrix} -J \\ Q^{(K)} \end{pmatrix} \quad (3.39)$$

Equations (3.35)-(3.38) are a set of differential equations which are decoupled from the bulk in the sense that they do not have any r -derivative term. They involve only functions of angular variables and respective derivatives. Solving these equations provides the information only on the horizon with no reference to the bulk of the space and in particular asymptotic values of the fields. This guarantees the decoupling of the dynamics of the horizon from the bulk. If these equations have unique solutions, this decoupling will prove the attractor mechanism in its strong form. It implies that the field configurations on the horizon are determined uniquely by these equations and the behavior at the infinity does not enter in specifying their values. If the equations admit more than one solution, the problem needs further analysis and one must explore which class of asymptotic conditions corresponds to a particular solution at the horizon [4].

It is notable that solutions of these equations determine $M(\theta)$, $N(\theta)$, $\Phi(\theta)$ and Σ which are required for specifying the near horizon geometry (2.16). We find them without using directly the equations in the near horizon. The boundary values for solving these equations are given by (3.22) and (3.23).

Our analysis clearly shows that this decoupling occurs for the double-horizon black holes and for distinct-horizon cases there is not such a decoupling. If we write the field equations for the black holes with the distinct horizons, a number of terms with r -derivative factors will survive. The presence of these terms obstructs the decoupling of the equations and the behavior of the fields will depend on the boundary conditions at infinity. One expects that as the distance between the inner and outer horizon decreases and the black hole approaches the double-horizon case, the decoupling violating terms become smaller and correspondingly less important.

3.1 Generalization to $f(R)$ Gravities

Our method and argument can be simply generalized to include $f(R)$ gravities. In these theories the term $\sqrt{-G}R$ in the action is replaced by the $\sqrt{-G}f(R)$ resulting in the action,

$$I = \frac{1}{\kappa^2} \int d^4x \sqrt{-G} f(R) \quad (3.40)$$

The other terms of the action remain unchanged and therefore we only study the effect of this term on the above considerations. General analysis of the section two which is independent of any particular action shows that the black hole solutions of this theory also have the same form as (3.2).

The simplest way of studying $f(R)$ gravities is using their equivalence to Einstein gravity

coupled to a scalar field. If we define σ by

$$\sigma = \frac{\sqrt{3}}{2} \ln \left| \kappa^2 \frac{df(R)}{dR} \right| \quad (3.41)$$

and make a conformal transformation,

$$g_{\mu\nu} = \left| \kappa^2 \frac{df(R)}{dR} \right| G_{\mu\nu} = \exp\left(\frac{2\sigma}{\sqrt{3}}\right) G_{\mu\nu}. \quad (3.42)$$

The field equations derived from (3.40) are equivalent to those derived from the action

$$\hat{S} = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left(R - 2g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V(\sigma) \right) \quad (3.43)$$

where

$$V = \lambda \exp\left(-\frac{4\sigma}{\sqrt{3}}\right) \left(R \frac{df(R)}{dR} - f(R) \right) \quad (3.44)$$

and

$$\lambda = \begin{cases} 1 & \text{if } \frac{df(R)}{dR} > 0 \\ -1 & \text{if } \frac{df(R)}{dR} < 0 \end{cases} \quad (3.45)$$

It is seen that $V(\sigma)$ is the Legendre transformation of $f(R)$.

Using (3.2) and (3.42) we can write $g_{\mu\nu}$ as

$$d\hat{s}^2 = \hat{A}(r, \theta) \left(-\frac{S(r)}{B(r, \theta)} dt^2 + \frac{1}{S(r)} dr^2 \right) + \hat{E}(r, \theta) d\theta^2 + \hat{F}(r, \theta) \left(d\phi + C(r, \theta) dt \right)^2 \quad (3.46)$$

where

$$\begin{aligned} \hat{A}(r, \theta) &= \exp\left(\frac{2\sigma}{\sqrt{3}}\right) A(r, \theta) \\ \hat{E}(r, \theta) &= \exp\left(\frac{2\sigma}{\sqrt{3}}\right) E(r, \theta) \\ \hat{F}(r, \theta) &= \exp\left(\frac{2\sigma}{\sqrt{3}}\right) F(r, \theta) \end{aligned} \quad (3.47)$$

Hence the problem is reduced to the previous case and we can repeat our method.

Reduction of this theory in ϕ direction results a theory with the action

$$\hat{S} = \frac{2\pi}{\kappa^2} \int d^3x \sqrt{-\hat{g}} \left(\hat{R} - 2\hat{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - 2\hat{g}^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - w(\Phi) F_{\mu\nu} F^{\mu\nu} - U(\Phi, \sigma) \right) \quad (3.48)$$

where

$$w(\Phi) = \frac{1}{4} e^{-4\Phi} \quad (3.49)$$

$$U(\Phi, \sigma) = e^{2\Phi} V(\sigma) \quad (3.50)$$

and

$$d\tilde{s}^2 = M(r, \theta) \left(-\frac{S(r)}{B(r, \theta)} dt^2 + \frac{1}{S(r)} dr^2 \right) + N(r, \theta) d\theta^2 \quad (3.51)$$

$$e^{-2\Phi} = \hat{F}(r, \theta) \quad (3.52)$$

$$A_t = C(r, \theta) \quad (3.53)$$

with

$$\begin{aligned} M(r, \theta) &= \hat{F}(r, \theta) \hat{A}(r, \theta) \\ N(r, \theta) &= \hat{F}(r, \theta) \hat{E}(r, \theta) \end{aligned} \quad (3.54)$$

Field equations from variation of (3.48) with respect to the metric, the scalars and the gauge field are given by

$$\hat{R}_{\mu\nu} - 2\partial_\mu\Phi\partial_\nu\Phi - 2\partial_\mu\sigma\partial_\nu\sigma = w(\Phi)(2F_{\mu\lambda}F_\nu{}^\lambda - \hat{g}_{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}) + \hat{g}_{\mu\nu}U(\Phi, \sigma) \quad (3.55)$$

$$\frac{1}{\sqrt{-\hat{g}}} \partial_\mu(\sqrt{-\hat{g}} \partial^\mu\Phi) = \frac{1}{4} \frac{\delta w(\Phi)}{\delta\Phi} F_{\mu\nu}F^{\mu\nu} + \frac{1}{4} \frac{\delta U(\Phi, \sigma)}{\delta\Phi} \quad (3.56)$$

$$\frac{1}{\sqrt{-\hat{g}}} \partial_\mu(\sqrt{-\hat{g}} \partial^\mu\sigma) = \frac{1}{4} \frac{\delta U(\Phi, \sigma)}{\delta\sigma} \quad (3.57)$$

$$\partial_\mu(\sqrt{-\hat{g}} w(\Phi) F^{\mu\nu}) = 0 \quad (3.58)$$

At the horizon of the double-horizon black holes these equations take the forms,

$$4N^2(\theta) + 2N(\theta)M''(\theta) - N'(\theta)M'(\theta) = -4N^2(\theta)M(\theta)U(\Phi, \sigma) \quad (3.59)$$

$$\begin{aligned} 4M(\theta)N(\theta) + \left(M'(\theta)\right)^2 &= 4M^2(\theta)(\Phi')^2 + 4M^2(\theta)(\sigma')^2 + 4\Sigma^2N(\theta)w(\Phi) \\ &\quad - 2N(\theta)M^2(\theta)U(\Phi, \sigma) \end{aligned} \quad (3.60)$$

$$\frac{M(\theta)}{\sqrt{N(\theta)}} \left(\frac{M(\theta)}{\sqrt{N(\theta)}} \Phi' \right)' = -\frac{1}{2} \Sigma^2 \frac{\delta w(\Phi)}{\delta\Phi} + \frac{1}{4} M^2(\theta) \frac{\delta U(\Phi, \sigma)}{\delta\Phi} \quad (3.61)$$

$$\frac{M(\theta)}{\sqrt{N(\theta)}} \left(\frac{M(\theta)}{\sqrt{N(\theta)}} \sigma' \right)' = \frac{1}{4} M^2(\theta) \frac{\delta U(\Phi, \sigma)}{\delta\sigma} \quad (3.62)$$

$$\int_0^\pi d\theta \frac{\sqrt{N(\theta)}}{M(\theta)} \Sigma w(\Phi) = -2J \quad (3.63)$$

The derivatives are with respect to θ and we have defined

$$\Sigma = \sqrt{B} \alpha \quad (3.64)$$

Again these equations are a set of decoupled equations from the bulk. Solving them gives $M(\theta)$, $N(\theta)$, $\Phi(\theta)$, $\sigma(\theta)$ and Σ which determine the near horizon geometry of the double-horizon black holes in $f(R)$ gravity.

We note that a special case is when $f(R) = R + \Lambda$ where Λ is a constant. In this case the black hole is asymptotically AdS . Hence our analysis is valid even in the presence of the cosmological constant.

4 Double-Horizon Limit and Entropy Function Method

Our Analysis in the previous sections clearly shows the decoupling of the dynamics at the horizon from the bulk. This decoupling occurs at the level of the equations of motion, but an interesting question is to see what happens at the level of the action. We consider this point in this section.

Let us consider charged rotating black holes in the theory of gravity which is described by the action (3.1). The form of the metric, gauge field and scalar field are given by (3.2), (3.4) and (3.5). A proper way to deal with this theory is the reduction in ϕ direction and studying the new three dimensional theory which is described by (3.14)-(3.19). At the horizon of the double-horizon black holes, field equations derived from the action of this new theory take the forms (3.35)-(3.38).

We define

$$L = R - H_{ij}(\Phi)\partial_\mu\Phi^i\partial^\mu\Phi^j - W_{ab}(\Phi)F^a_{\mu\nu}F^{b\mu\nu} - U(\Phi) \quad (4.1)$$

thus from (3.14) it is seen that

$$S = \frac{1}{8} \int d^3x \sqrt{-g} L \quad (4.2)$$

where we have chosen $\kappa = 16\pi$.

By using the double-horizon condition, at the horizon we have

$$\begin{aligned} l(\theta) \equiv L \Big|_{r=r_H} = & \frac{1}{2 N^2(\theta) M^2(\theta)} \left(N(\theta) (M'(\theta))^2 + 2 M(\theta) N'(\theta) M'(\theta) - 4 N(\theta) M(\theta) M''(\theta) \right. \\ & - 4 N^2(\theta) M(\theta) - 2 N(\theta) M^2(\theta) (\Phi')^T \mathbf{H} \Phi' \\ & \left. + 4 N^2(\theta) \Sigma^T \mathbf{W} \Sigma \right) - U(\Phi) \end{aligned} \quad (4.3)$$

It is obvious that $l(\theta)$ does not have any r-derivative term and it is decoupled. Due to this decoupled property of $l(\theta)$, equations (3.35)-(3.38) are given respectively as

$$\frac{\delta f}{\delta M(\theta)} = 0 \quad (4.4)$$

$$\frac{\delta f}{\delta N(\theta)} = 0 \quad (4.5)$$

$$\frac{\delta f}{\delta \Phi_i} = 0 \quad (4.6)$$

$$\frac{\delta f}{\delta \alpha_i} = Q_i \quad (4.7)$$

where

$$f = \frac{1}{8} \int d\theta \sqrt{-g} l(\theta) \quad (4.8)$$

Generating equations of motion by extremizing a function which is defined at the horizon, is similar to the *Entropy Function Method* which determines parameters of the near-horizon geometry [7],[8]. The basic point here is that in our case this extremization is a direct consequence of the double-horizon limit. It is not difficult to see that F which is defined as

$$F = \alpha_i Q^i - f \quad (4.9)$$

is exactly the entropy function if we started from the near horizon geometry.

Our analysis shows that why we can not provide a method like the entropy function method for the black holes with distinct horizons. For these cases it is not possible to cancel r-derivative terms of the action at the horizon and thus we do not have a decoupled action which gives the equations of motion.

Using equations (4.4)-(4.7) we can simplify Wald's entropy formula. We follow the method of [14]. The entropy of the black hole is given by

$$S_{BH} = -8\pi \sqrt{-h} \frac{\partial \mathcal{L}^{(2)}}{\partial R_{rtrt}^{(2)}} \sqrt{-h_{rr} h_{tt}} \quad (4.10)$$

where $h_{\alpha\beta}$ with $\alpha, \beta = r, t$ is a two dimensional metric defined as

$$h_{\alpha\beta} = \frac{1}{2} \int_0^\pi d\theta \sin\theta g_{\alpha\beta} \quad (4.11)$$

and $\sqrt{-h} \mathcal{L}^{(2)}$ is the two dimensional Lagrangian density, related to the three dimensional Lagrangian density via the formula:

$$\sqrt{-h} \mathcal{L}^{(2)} = \int d\theta \sqrt{-g} \mathcal{L} \quad (4.12)$$

It follows from (3.15) and (4.11) that at the horizon, after imposing double-horizon limit we have

$$\sqrt{-h_{rr}h_{tt}} = \sqrt{B} R_{trt}^{(2)} \quad (4.13)$$

thus we can express (4.10) at the horizon as

$$S_{BH} = -8\pi \sqrt{B} \sqrt{-h} \frac{\partial \mathcal{L}^{(2)}}{\partial R_{trt}^{(2)}} R_{trt}^{(2)} \quad (4.14)$$

Following the method of [14], one can use the equations (4.4)-(4.7) to simplify (4.14) and obtain

$$S_{BH} = 2\pi \sqrt{B} \left(\alpha^T \mathbf{Q} - f \right) \quad (4.15)$$

where f is evaluated at the extremized values. This is the same result as the entropy function method. The extra factor \sqrt{B} is a result of this point that we did not use near-horizon geometry. We can cancel it in the first term by the definition of Σ and in the second term by $\frac{1}{\sqrt{B}}$ factor of $\sqrt{-g}$, hence it does not enter in the calculations.

5 Conclusion

The analysis provided in this paper has two aspects. First it gives a deeper physical reason for AdS part of the near horizon geometry of the double-horizon (extremal) black holes which has been the beginning point of the entropy function method. Second it puts our earlier result about the decoupling of the dynamics of the horizon for double-horizon black holes on a firm ground. It also opens venues for further investigation of its properties.

Our analysis also clarify that why black holes with distinct horizons do not enjoy a decoupling or attractor mechanism. The nature of approach to the decoupling limit is also of interest. The decoupling given as a set of equations on the compact sphere of the horizon which can be solved consistently. If it has a unique solution then the attractor mechanism works like the non rotating case. Even if the solution is not unique we expect it to result in a discrete set of solutions closely related to the minima of the potential in the non-rotating case. The relation of different solutions of such discrete set to the large distance boundary condition and quantum transition between these solution are not clear yet.

Solution to the set of the decoupled equations of the horizon provides sufficient information for the physical properties of the black hole in particular the entropy. Hence we may find a way to understanding of the fact that all the information hidden in a black hole is distributed on the surface of the horizon.

The other interesting direction for further investigation is to generalize this method to the higher dimensions where the topology of the horizon is more complicated than a simple sphere.

These questions and other unclear properties of the double-horizon limit is under investigation.

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